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COMBINATORIAL CONVOLUTION SUMS DERIVED FROM DIVISOR FUNCTIONS AND FAULHABER SUMS

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Abstract. It is known that certain convolution sums using Liouville identity can be expressed as a combination of divisor functions and Bernoulli numbers. In this article we find seven combinatorial convolution sums derived from divisor functions and Bernoulli numbers.

1. Introduction

The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{C}$ denote the set of natural numbers, the ring of integers the field of rational numbers and the field of complex numbers, respectively.

The Bernoulli polynomials $B_k(x)$, which are usually defined by the exponential generating function

$$\frac{te^{xt}}{e^t-1} = \sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!},$$

play an important role in different areas of mathematics including number theory and the theory of finite differences. The Bernoulli polynomials satisfy the following well-known identities :

$$\begin{aligned}\sum_{j=0}^{N} j^k &= \frac{B_{k+1}(N+1)-B_{k+1}(0)}{k+1}, \qquad (k \geq 1)\\ &= \frac{1}{k+1}\sum_{j=0}^{k}(-1)^j\binom{k+1}{j}B_jN^{k+1-j}.\end{aligned}$$

The Bernoulli numbers B_k are defined to be $B_k := B_k(0)$.

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For $n \in \mathbb{N}$, $k \in \mathbb{Z}_{\geq 0}$, and $l \in \mathbb{Q}$, we define some divisor functions:

$$\sigma_k(n) := \sum_{d|n} d^k, \quad \sigma_k^*(n) := \sum_{\substack{d|n \\ \frac{n}{d}\, odd}} d^k,$$
$$\sigma_k^{\#}(n) := \sum_{\substack{d|n \\ 2\nmid d}} d^k, \quad \hat{\sigma}_{k,l}(n) := \sigma_k(n) - l\sigma_k\Big(\frac{n}{2}\Big),$$
$$H_k(N) := \sum_{j=0}^{N} j^k, \quad \hat{H}_{k,l}(n) := H_k(n) - lH_k(n/2).$$

It is clear that

$$\hat{\sigma}_{k,0}(n) = \sigma_k(n), \quad \hat{\sigma}_{k,1}(n) = \sigma_k^*(n) = \sigma_k(n) - \sigma_k(n/2), \quad \hat{\sigma}_{k,2^k}(n) = \widetilde{\sigma}_k(n)$$

and

$$\sum_{d|n} \hat{H}_{k,l}(d) = \frac{1}{k+1} \sum_{j=0}^{k} (-1)^j \binom{k+1}{j} B_j \hat{\sigma}_{k+1-j,l}(n).$$

The identity

$$\sum_{k=1}^{n-1} \sigma(k)\sigma(n-k) = \frac{5}{12}\sigma_3(n) + \left(\frac{1}{12} - \frac{1}{2}n\right)\sigma(n)$$

for the basic convolution sum first appeared in a letter from Besge to Liouville in 1862 (see [2]). For some of the history of the subject, and for a selection of these articles, we mention [3, 9, 10], and especially [5, 11]. The study of convolution sums and their applications is classic and they play an important role in number theory. In this paper we are trying to focus on the combinatorial convolution sums. For positive integers k and n, the combinatorial convolution sum

$$\sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{n-1} \hat{\sigma}_{2k-2s-1,l}(m)\hat{\sigma}_{2s+1,l}(n-m) \tag{1.1}$$

can be evaluated explicitly in terms of divisor functions and a sum of involving Faulhaber sums. We are motivated by Ramanujan's recursion formula for sums of the product of two Eisenstein series [1, Entry 14, p. 332] and its proof, and also the following propositions.

PROPOSITION 1.1 ([11]). *Let k, n be positive integers. Then*

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-m)$$
$$=\frac{2k+3}{4k+2}\sigma_{2k+1}(n)+\left(\frac{k}{6}-n\right)\sigma_{2k-1}(n)$$
$$+\frac{1}{2k+1}\sum_{j=2}^{k}\binom{2k+1}{2j}B_{2j}\sigma_{2k+1-2j}(n),$$

where B_j is the j-th Bernoulli number.

PROPOSITION 1.2 ([4]). *For any integers $k \geq 1$ and $N \geq 3$, we have*

$$\sum_{r=0}^{2k}\binom{2k}{r}\sum_{i=1}^{\left[\frac{N-1}{2}\right]}\sum_{m=1}^{n-1}\sigma_{2k-r}(m;i,N)\sigma_r(n-m;i,N)$$
$$=\sigma^*_{2k+1}(n;N)-\frac{2}{N}n\sigma^*_{2k-1}(n;N)-\frac{1}{N}\sum_{i=1}^{\left[\frac{N-1}{2}\right]}(N-2i)\sigma_{2k}(n;i,N)$$
$$-\frac{1+(-1)^N}{2}\left(\sigma^*_{2k+1}\left(\frac{2n}{N};2\right)-\frac{2}{N}n\sigma^*_{2k-1}\left(\frac{2n}{N};2\right)\right),$$

where

$$\sigma_r(n;i,N)=\sum_{\substack{d|n\\ \frac{n}{d}\equiv i(N)}}d^r-(-1)^r\sum_{\substack{d|n\\ \frac{n}{d}\equiv -i(N)}}d^r$$

and

$$\sigma^*_r(n;N)=\sum_{\substack{d|n\\ \frac{n}{d}\not\equiv 0(N)}}d^r=\sigma_r(n)-\sigma_r(n/N).$$

The aim of this article is to study seven combinatorial convolution sums of the analogous type (1.1). More precisely, we prove the following theorems and corollaries.

THEOREM 1.3. *Let k, n be positive integers, and let l be a rational number. Then*

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\hat{\sigma}_{2k-2s-1,l}(m)\hat{\sigma}_{2s+1,l}(n-m)$$
$$=\frac{1}{2}\hat{\sigma}_{2k+1,-l^2+2l}(n)-\frac{1-l}{2}\hat{\sigma}_{2k,l}(n)\tag{1.2}$$
$$-\frac{n(2-l)}{2}\hat{\sigma}_{2k-1,l}(n)+(1-l)\sum_{d|n}\hat{H}_{2k,l}(d).$$

REMARK 1.4. In Theorem 1.3, $l=0$ and $l=1$ recover Proposition 1.1 and Proposition 2.2, respectively.

For $k=1$, we have the following corollary.

COROLLARY 1.5. *For any $n \in \mathbb{N}$ and $l \in \mathbb{Q}$, we have*

$$\sum_{m=1}^{n-1} \hat{\sigma}_{1,l}(m)\hat{\sigma}_{1,l}(n-m) = \frac{1}{4}\hat{\sigma}_{3,-l^2+2l}(n) + \frac{1-l}{6}\hat{\sigma}_{3,l}(n) + \frac{3n(l-2)+1-l}{12}\hat{\sigma}_{1,l}(n).$$

REMARK 1.6. In the preceding corollary, $l=2$ recovers [7, (11)].

COROLLARY 1.7. *Let k,n be positive integers. Then*

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\sigma_{2k-2s-1}(m/2)\sigma_{2s+1}(n-m)$$
$$= \frac{1}{2}\sigma_{2k+1}(n/2) - \frac{1}{4}\hat{\sigma}_{2k,-1}(n) - \frac{n}{4}\hat{\sigma}_{2k-1,-2}(n) + \frac{1}{2}\sum_{d|n}\hat{H}_{2k,-1}(d).$$

COROLLARY 1.8. *Let k,n be positive integers. Then*

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\sigma^*_{2k-2s-1}(m)\sigma_{2s+1}(n-m)$$
$$= \frac{1}{2}\sigma^*_{2k+1}(n) - \frac{1}{4}\sigma^*_{2k}(n) - \frac{3n}{4}\hat{\sigma}_{2k-1,\frac{2}{3}}(n) + \frac{1}{2}\sum_{d|n}\hat{H}_{2k}(d).$$

THEOREM 1.9. *Let k,n be positive integers and let l be a rational number. Then*

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\hat{\sigma}_{2k-2s-1,l}(2m)\hat{\sigma}_{2s+1,l}(2n-2m)$$
$$\begin{aligned} &= \frac{1-l}{4}\hat{\sigma}_{2k+1,2l-1}(2n) - \frac{1-l}{2}\hat{\sigma}_{2k,l}(2n) - 2n(1-l)\hat{\sigma}_{2k-1,l/2}(2n) \\ &\quad + 2^{2k-1}l(\sigma^*_{2k+1}(n) - n\sigma^*_{2k-1}(n)) + (1-l)\sum_{d|2n}\hat{H}_{2k,l}(d) \end{aligned} \tag{1.3}$$

and

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n}\hat{\sigma}_{2k-2s-1,l}(2m-1)\hat{\sigma}_{2s+1,l}(2n-2m+1)$$
$$\begin{aligned} &= \frac{1}{2}\hat{\sigma}_{2k+1,l}(2n) + \frac{l(l-1)}{2}\sigma_{2k+1}(n) + n(l-2)\hat{\sigma}_{2k-1,l}(2n) \\ &\quad + \frac{l-1}{4}\hat{\sigma}_{2k+1,2l-1}(2n) - 2n(l-1)\hat{\sigma}_{2k-1,l/2}(2n) \\ &\quad - 2^{2k-1}l(\sigma^*_{2k+1}(n) - n\sigma^*_{2k-1}(n)). \end{aligned} \tag{1.4}$$

REMARK 1.10. In (1.3), $l = 0$ and $l = 1$ recover Proposition 2.1 and Proposition 2.2, respectively. And in (1.4), $l = 0$ recovers [8, Lemma 6.2].

If we insert $k = 1$ in the preceding theorem, one has the following corollary.

COROLLARY 1.11. *For $n \in \mathbb{N}$ and $l \in \mathbb{Q}$ we have*

$$\begin{aligned}\sum_{m=1}^{n-1} &\hat{\sigma}_{1,l}(2m)\hat{\sigma}_{1,l}(2n-2m)\\ &= \frac{1-l}{8}\hat{\sigma}_{3,2l-1}(2n) + \frac{1-l}{6}\hat{\sigma}_{3,l}(2n) - n(1-l)\hat{\sigma}_{1,l/2}(2n)\\ &\quad + \frac{1-l}{12}\hat{\sigma}_{1,l}(2n) + l(\sigma_3^*(n) - n\sigma_1^*(n))\end{aligned}$$

and

$$\begin{aligned}\sum_{m=1}^{n} &\hat{\sigma}_{1,l}(2m-1)\hat{\sigma}_{1,l}(2n-2m+1)\\ &= \frac{1}{4}\hat{\sigma}_{3,l}(2n) + \frac{l(l-1)}{4}\sigma_3(n) + \frac{n(l-2)}{2}\hat{\sigma}_{1,l}(2n)\\ &\quad + \frac{l-1}{8}\hat{\sigma}_{3,2l-1}(2n) - n(l-1)\hat{\sigma}_{1,l/2}(2n) - l(\sigma_3^*(n) - n\sigma_1^*(n)).\end{aligned}$$

THEOREM 1.12. *Let k, n be positive integers. Then*

$$\begin{aligned}\sum_{s=0}^{k-1}\binom{2k}{2s+1}&\sum_{m=1}^{n-1} 2^{2k-2s-1}\sigma_{2k-2s-1}(m/4)\sigma_{2s+1}(n-m)\\ &= \frac{1}{4}\sigma_{2k+1}(n/2) - \frac{1}{4}(\sigma_{2k}(n) - 2^{2k+1}\sigma_{2k}(n/2) - 2^{2k}\sigma_{2k}(n/4))\\ &\quad - \frac{n}{4}(\sigma_{2k-1}(n) + 2^{2k}\sigma_{2k-1}(n/4)) + \frac{1}{2}\sigma_{2k}^{\#}(n)\\ &\quad + \frac{2^{2k-1}}{2k+1}\sum_{j=0}^{2k}\binom{2k+1}{j}B_j\sigma_{2k+1-j}(n/2)\\ &\quad + \frac{1}{2(2k+1)}\sum_{j=0}^{2k}\binom{2k+1}{j}B_j 2^{2k+1-j}\sigma_{2k+1-j}(n/4)\\ &\quad + \frac{1}{2(2k+1)}\sum_{j=0}^{2k}\binom{2k+1}{j}B_j\sigma_{2k+1-j}^{\#}(n)\\ &\quad - \frac{1}{2(2k+1)}\sum_{u+v+w=2k+1} 2^{v-1}\binom{2k+1}{u,v,w}B_v\sigma_w^{\#}(n).\end{aligned}$$

where $\binom{2k+1}{u,v,w} = \frac{(2k+1)!}{u!v!w!}$.

Remark 1.13. In the preceding theorem, $k = 1$ recovers [11, Theorem 15.2].

2. Proofs of the theorems and corollaries

To prove the theorems and corollaries, we need the following propositions.

Proposition 2.1 ([8, Theorem 6.3]). *Let k, n be positive integers. Then*

$$\begin{aligned}&\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\sigma_{2k-2s-1}(2m)\sigma_{2s+1}(2n-2m)\\&\quad=\frac{2k+3}{4k+2}\sigma_{2k+1}(2n)-\frac{1}{4}\sigma^*_{2k+1}(2n)+\left(\frac{k}{6}-2n\right)\sigma_{2k-1}(2n)\\&\quad\quad+\frac{1}{2k+1}\sum_{j=2}^{k}\binom{2k+1}{2j}B_{2j}\sigma_{2k+1-2j}(2n).\end{aligned}$$

Proposition 2.2 ([6, Identity (10)]). *Let k, n be positive integers. Then*

$$\begin{aligned}&\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\sigma^*_{2k-2s-1}(m)\sigma^*_{2s+1}(n-m)\\&\quad=\frac{1}{2}(\sigma^*_{2k+1}(n)-n\sigma^*_{2k-1}(n)).\end{aligned}$$

Proposition 2.3 ([11, p.172]). *Let n be a positive integer. Let $f:\mathbb{Z}\to\mathbb{C}$ be an even function. Then*

$$\begin{aligned}&\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\4ax+by=n}}(f(2a-b)-f(2a+b))\\&=\frac{1}{2}f(0)(\sigma(n/2)-d(n/2)-d(n/4))+\frac{1}{2}\sum_{d|n}\left(1+\frac{n}{d}\right)f(d)\\&\quad-\frac{1}{2}\sum_{d|n/2}df(d)+\frac{1}{2}\sum_{d|n/4}\left(1+\frac{n}{d}\right)f(2d)-\sum_{d|n}\sum_{\substack{l=1\\l\equiv d(2)}}^{d}f(l)-\sum_{d|n/4}\sum_{l=1}^{2d}f(l).\end{aligned}$$

PROOF OF THEOREM 1.3. Let $k, n \in \mathbb{N}$ and $l \in \mathbb{Q}$. Then the left hand side is

$$\begin{aligned}
&\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\hat{\sigma}_{2k-2s-1,l}(m)\hat{\sigma}_{2s+1,l}(n-m)\\
&\quad=\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\Big(\sigma_{2k-2s-1}(m)-l\sigma_{2k-2s-1}\Big(\frac{m}{2}\Big)\Big)\\
&\qquad\Big(\sigma_{2s+1}(n-m)-l\sigma_{2s+1}\Big(\frac{n-m}{2}\Big)\Big)\\
&\quad=\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\Big(\sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-m)\\
&\qquad+l^2\sigma_{2k-2s+1}\Big(\frac{m}{2}\Big)\sigma_{2s+1}\Big(\frac{n-m}{2}\Big)\Big)\\
&\qquad-l\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\Big(\sigma_{2k-2s-1}(m)\sigma_{2s+1}\Big(\frac{n-m}{2}\Big)\\
&\qquad+\sigma_{2k-2s+1}\Big(\frac{m}{2}\Big)\sigma_{2s+1}(n-m)\Big)\\
&\quad=l\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\Big(\sigma_{2k-2s-1}(m)-\sigma_{2k-2s-1}\Big(\frac{m}{2}\Big)\Big)\\
&\qquad\Big(\sigma_{2s+1}(n-m)-\sigma_{2s+1}\Big(\frac{n-m}{2}\Big)\Big)\\
&\qquad+(1-l)\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\Big(\sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-m)\\
&\qquad-l\sigma_{2k-2s-1}\Big(\frac{m}{2}\Big)\sigma_{2s+1}\Big(\frac{n-m}{2}\Big)\Big).
\end{aligned}\tag{2.1}$$

Now the first summation of the right side of (2.1) becomes

$$\begin{aligned}
&\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\Big(\sigma_{2k-2s-1}(m)-\sigma_{2k-2s-1}\Big(\frac{m}{2}\Big)\Big)\\
&\qquad\Big(\sigma_{2s+1}(n-m)-\sigma_{2s+1}\Big(\frac{n-m}{2}\Big)\Big)\\
&\quad=\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\sigma^*_{2k-2s-1}(m)\sigma^*_{2s+1}\Big(\frac{n-m}{2}\Big)\\
&\quad=\frac{1}{2}(\sigma^*_{2k+1}(n)-n\sigma^*_{2k-1}(n))
\end{aligned}\tag{2.2}$$

by Proposition 2.2. Consider the second summation of the right side of (2.1) by the use of Proposition 1.1. It equals

$$
\begin{aligned}
&\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-m)\\
&\quad=\frac{2k+3}{4k+2}\sigma_{2k+1}(n)+\left(\frac{k}{6}-n\right)\sigma_{2k-1}(n)\\
&\qquad+\frac{1}{2k+1}\sum_{j=2}^{k}\binom{2k+1}{2j}B_{2j}\sigma_{2k+1-2j}(n)\\
&\quad=\frac{1}{2}\sigma_{2k+1}(n)-\frac{1}{2}\sigma_{2k}(n)-n\sigma_{2k-1}(n)\\
&\qquad+\frac{1}{2k+1}\sum_{j=0}^{2k}\binom{2k+1}{j}B_{j}\sigma_{2k+1-j}(n)\\
&\quad=\frac{1}{2}\sigma_{2k+1}(n)-\frac{1}{2}\sigma_{2k}(n)-n\sigma_{2k-1}(n)+\sum_{d|n}H_{2k}(d).
\end{aligned}
\tag{2.3}
$$

Using Proposition 1.1, we observe that last summation of the right side of (2.1) is

$$
\begin{aligned}
&\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\sigma_{2k-2s-1}\left(\frac{m}{2}\right)\sigma_{2s+1}\left(\frac{n-m}{2}\right)\\
&\quad=\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{[\frac{n}{2}]-1}\sigma_{2k-2s-1}(m)\sigma_{2s+1}\left(\frac{n}{2}-m\right)\\
&\quad=\frac{2k+3}{4k+2}\sigma_{2k+1}\left(\frac{n}{2}\right)+\left(\frac{k}{6}-\frac{n}{2}\right)\sigma_{2k-1}\left(\frac{n}{2}\right)\\
&\qquad+\frac{1}{2k+1}\sum_{j=2}^{k}\binom{2k+1}{2j}B_{2j}\sigma_{2k+1-2j}\left(\frac{n}{2}\right)\\
&\quad=\frac{1}{2}\sigma_{2k+1}\left(\frac{n}{2}\right)-\frac{1}{2}\sigma_{2k}\left(\frac{n}{2}\right)-\frac{n}{2}\sigma_{2k-1}\left(\frac{n}{2}\right)\\
&\qquad+\frac{1}{2k+1}\sum_{j=0}^{2k}\binom{2k+1}{j}B_{j}\sigma_{2k+1-j}\left(\frac{n}{2}\right)\\
&\quad=\frac{1}{2}\sigma_{2k+1}\left(\frac{n}{2}\right)-\frac{1}{2}\sigma_{2k}\left(\frac{n}{2}\right)-\frac{n}{2}\sigma_{2k-1}\left(\frac{n}{2}\right)+\sum_{d|n}H_{2k}\left(\frac{d}{2}\right).
\end{aligned}
\tag{2.4}
$$

Using (2.2), (2.3) and (2.4), we get

$$\begin{aligned}
&\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\hat{\sigma}_{2k-2s-1,l}(m)\hat{\sigma}_{2s+1,l}(n-m)\\
&\quad=\frac{l}{2}(\sigma^*_{2k+1}(n)-n\sigma^*_{2k-1}(n))\\
&\qquad+(1-l)\Big(\frac{1}{2}\sigma_{2k+1}(n)-\frac{1}{2}\sigma_{2k}(n)-n\sigma_{2k-1}(n)+\sum_{d|n}H_{2k}(d)\Big)\\
&\qquad-l(1-l)\Big(\frac{1}{2}\sigma_{2k+1}\Big(\frac{n}{2}\Big)-\frac{1}{2}\sigma_{2k}\Big(\frac{n}{2}\Big)-\frac{n}{2}\sigma_{2k-1}\Big(\frac{n}{2}\Big)+\sum_{d|n}H_{2k}\Big(\frac{d}{2}\Big)\Big)\\
&\quad=\frac{1}{2}\sigma_{2k+1}(n)-\frac{l(2-l)}{2}\sigma_{2k+1}\Big(\frac{n}{2}\Big)-\frac{1-l}{2}\Big(\sigma_{2k}(n)-l\sigma_{2k}\Big(\frac{n}{2}\Big)\Big)\\
&\qquad+\frac{n(l-2)}{2}\Big(\sigma_{2k-1}(n)-l\sigma_{2k-1}\Big(\frac{n}{2}\Big)\Big)-(l-1)\sum_{d|n}\hat{H}_{2k,l}(d)\\
&\quad=\frac{1}{2}\hat{\sigma}_{2k+1,l}(n)-\frac{l(1-l)}{2}\sigma_{2k+1}\Big(\frac{n}{2}\Big)-\frac{1-l}{2}\hat{\sigma}_{2k,l}(n)\\
&\qquad-\frac{n(2-l)}{2}\hat{\sigma}_{2k-1,l}(n)+(1-l)\sum_{d|n}\hat{H}_{2k,l}(d).
\end{aligned}$$

Therefore the proof is completed. ∎

Proof of Corollary 1.7. We take $l=-1$ in Theorem 1.3. Then the left hand side of Theorem 1.3 is

$$\begin{aligned}
\mathfrak{L}=\sum_{s=0}^{k-1}\binom{2k}{2s+1}&\sum_{m=1}^{n-1}(\sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-m)\\
&+\sigma_{2k-2s-1}(m/2)\sigma_{2s+1}((n-m)/2)\\
+\sum_{s=0}^{k-1}\binom{2k}{2s+1}&\sum_{m=1}^{n-1}(\sigma_{2k-2s-1}(m/2)\sigma_{2s+1}(n-m)\\
&+\sigma_{2k-2s-1}(m)\sigma_{2s+1}((n-m)/2).
\end{aligned}$$

Observing that replacing m and s by $n-m$ and $k-s$ respectively in $\mathfrak{L}$,

$$\begin{aligned}
&\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}(\sigma_{2k-2s-1}(m/2)\sigma_{2s+1}(n-m)\\
&\quad=\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=0}^{n-1}\sigma_{2k-2s-1}(m)\sigma_{2s+1}((n-m)/2),
\end{aligned}$$

we find

$$\begin{aligned}\mathfrak{L} = \sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}(&\sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-m)\\ &+\sigma_{2k-2s-1}(m/2)\sigma_{2s+1}((n-m)/2)\\ &+2\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\sigma_{2k-2s-1}(m/2)\sigma_{2s+1}(n-m).\end{aligned}$$

Consider the right hand side of Theorem 1.3. It it follows that

$$\frac{1}{2}\hat{\sigma}_{2k+1,-1}(n)+\sigma_{2k+1}(n/2)-\hat{\sigma}_{2k,-1}(n)-\frac{3n}{2}\hat{\sigma}_{2k-1,-1}(n)+2\sum_{d|n}\hat{H}_{2k,-1}(d).$$

By Proposition 1.1 and Theorem 1.3, we deduce that

$$\begin{aligned}2\sum_{s=0}^{k-1}&\binom{2k}{2s+1}\sum_{m=1}^{n-1}\sigma_{2k-2s-1}(m/2)\sigma_{2s+1}(n-m)\\ &=\frac{1}{2}\hat{\sigma}_{2k+1,-1}(n)+\sigma_{2k+1}(n/2)-\hat{\sigma}_{2k,-1}(n)-\frac{3n}{2}\hat{\sigma}_{2k-1,-1}(n)\\ &\quad+2\sum_{d|n}\hat{H}_{2k,-1}(d)-\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}(\sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-m)\\ &\qquad+\sigma_{2k-2s-1}(m/2)\sigma_{2s+1}((n-m)/2)\\ &=\frac{1}{2}\hat{\sigma}_{2k+1,-1}(n)+\sigma_{2k+1}(n/2)-\hat{\sigma}_{2k,-1}(n)-\frac{3n}{2}\hat{\sigma}_{2k-1,-1}(n)\\ &\quad+2\sum_{d|n}\hat{H}_{2k,-1}(d)-\Big(\frac{1}{2}\sigma_{2k+1}(n)-\frac{1}{2}\sigma_{2k}(n)-n\sigma_{2k-1}(n)+\sum_{d|n}H_{2k}(d)\\ &\quad+\frac{1}{2}\sigma_{2k+1}\Big(\frac{n}{2}\Big)-\frac{1}{2}\sigma_{2k}\Big(\frac{n}{2}\Big)-\frac{n}{2}\sigma_{2k-1}\Big(\frac{n}{2}\Big)+\sum_{d|n}H_{2k}\Big(\frac{d}{2}\Big)\Big)\\ &=\sigma_{2k+1}(n/2)-\frac{1}{2}\hat{\sigma}_{2k,-1}(n)-\frac{n}{2}\hat{\sigma}_{2k-1,-2}(n)+\sum_{d|n}\hat{H}_{2k,-1}(d).\end{aligned}$$

This completes the proof. □

Proof of Corollary 1.8. Let $k, n \in \mathbb{N}$. Then

$$\begin{aligned}\sum_{s=0}^{k-1}&\binom{2k}{2s+1}\sum_{m=1}^{n-1}\sigma^*_{2k-2s-1}(m)\sigma_{2s+1}(n-m)\\ &=\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\hat{\sigma}_{2k-2s-1,1}(m)\sigma_{2s+1}(n-m)\end{aligned}$$

$$= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{n-1} \sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-m)$$
$$- \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{n-1} \sigma_{2k-2s-1}(m/2)\sigma_{2s+1}(n-m).$$

Appealing to Corollary 1.7, we obtain

$$\sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{n-1} \sigma^*_{2k-2s-1}(m)\sigma_{2s+1}(n-m)$$
$$= \frac{1}{2}\sigma^*_{2k+1}(n) - \frac{1}{4}\sigma^*_{2k}(n) - \frac{n}{2}\sigma^*_{2k-1}(n) - \frac{n}{4}\sigma_{2k-1}(n) + \frac{1}{2}\sum_{d|n} \hat{H}_{2k}(d).$$

□

REMARK 2.4. Using Corollary 1.7, Corollary 1.8 and Proposition 2.2, we deduce

$$\sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{n-1} \hat{\sigma}_{2k-2s-1,2}(m)\sigma_{2s+1}(n-m)$$
$$= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{n-1} \sigma^*_{2k-2s-1}(m)\sigma^*_{2s+1}(n-m)$$

for n odd.

PROOF OF THEOREM 1.9. Let $k, n \in \mathbb{N}$ and $l \in \mathbb{Q}$. The left hand side of Theorem 1.9 is equal to

$$\sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{n-1} \hat{\sigma}_{2k-2s-1,l}(2m)\hat{\sigma}_{2s+1,l}(2n-2m)$$
$$= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{n-1} (\sigma_{2k-2s-1}(2m)\sigma_{2s+1}(2n-2m)$$
$$+ l^2\sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-m))$$
$$- l\sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{n-1} (\sigma_{2k-2s-1}(2m)\sigma_{2s+1}(n-m)$$
$$+ \sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-m))$$

$$= (1-l)\sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{n-1} (\sigma_{2k-2s-1}(2m)\sigma_{2s+1}(2n-2m)$$
$$- l\sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-m))$$

$$
\begin{aligned}
&+ l \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{n-1} (\sigma_{2k-2s-1}(2m) - \sigma_{2k-2s-1}(m)) \\
&\qquad (\sigma_{2s+1}(2n-2m) - \sigma_{2s+1}(n-m)) \\
&= (1-l) \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{n-1} (\sigma_{2k-2s-1}(2m)\sigma_{2s+1}(2n-2m) \\
&\qquad - l\sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-m)) \\
&\quad + l \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{n-1} \sigma^*_{2k-2s-1}(2m)\sigma^*_{2s+1}(2n-2m).
\end{aligned}
$$

We can divide it into three parts:

$$\sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{n-1} \sigma_{2k-2s-1}(2m)\sigma_{2s+1}(2n-2m), \tag{2.5}$$

$$\sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{n-1} \sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-m), \tag{2.6}$$

$$\sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{n-1} \sigma^*_{2k-2s-1}(2m)\sigma^*_{2s+1}(2n-2m). \tag{2.7}$$

By making use of Proposition 2.1, (2.5) can be written as

$$
\begin{aligned}
&\sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{n-1} \sigma_{2k-2s-1}(2m)\sigma_{2s+1}(2n-2m) \\
&\quad = \frac{2k+3}{4k+2}\sigma_{2k+1}(2n) - \frac{1}{4}\sigma^*_{2k+1}(2n) + \left(\frac{k}{6} - 2n\right)\sigma_{2k-1}(2n) \\
&\qquad + \frac{1}{2k+1} \sum_{j=2}^{k} \binom{2k+1}{2j} B_{2j}\sigma_{2k+1-2j}(2n) \\
&\quad = \frac{1}{2}\sigma_{2k+1}(2n) - \frac{1}{4}\sigma^*_{2k+1}(2n) - \frac{1}{2}\sigma_{2k}(2n) - 2n\sigma_{2k-1}(2n) + \sum_{d|2n} H_{2k}(d).
\end{aligned}
$$

Also, by Proposition 1.1, (2.6) equals

$$\begin{aligned}
&\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\sigma_{2k-2s-1}(m)\sigma_{2s+1}(n-m)\\
&\qquad=\frac{2k+3}{4k+2}\sigma_{2k+1}(n)+\left(\frac{k}{6}-n\right)\sigma_{2k-1}(n)\\
&\qquad\quad+\frac{1}{2k+1}\sum_{j=2}^{k}\binom{2k+1}{2j}B_{2j}\sigma_{2k+1-2j}(n)\\
&\qquad=\frac{1}{2}\sigma_{2k+1}(n)-\frac{1}{2}\sigma_{2k}(n)-n\sigma_{2k-1}(n)+\sum_{d|n}H_{2k}(d).
\end{aligned}$$

By Proposition 2.2, (2.7) is

$$\begin{aligned}
&\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\sigma^*_{2k-2s-1}(2m)\sigma^*_{2s+1}(2n-2m)\\
&\qquad=2^{2k}\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\sigma^*_{2k-2s-1}(m)\sigma^*_{2s+1}(n-m)\\
&\qquad=2^{2k-1}(\sigma^*_{2k+1}(n)-n\sigma^*_{2k-1}(n)).
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\hat{\sigma}_{2k-2s-1,l}(2m)\hat{\sigma}_{2s+1,l}(2n-2m)\\
&\quad=(1-l)\Big(\frac{1}{2}\sigma_{2k+1}(2n)-\frac{1}{4}\sigma^*_{2k+1}(2n)-\frac{1}{2}\sigma_{2k}(2n)\\
&\qquad\qquad-2n\sigma_{2k-1}(2n)+\sum_{d|2n}H_{2k}(d)\Big)\\
&\qquad-l(1-l)\Big(\frac{1}{2}\sigma_{2k+1}(n)-\frac{1}{2}\sigma_{2k}(n)-n\sigma_{2k-1}(n)+\sum_{d|n}H_{2k}(d)\Big)\\
&\qquad+2^{2k-1}l(\sigma^*_{2k+1}(n)-n\sigma^*_{2k-1}(n))\\
&\quad=\frac{1-l}{4}\hat{\sigma}_{2k+1,2l-1}(2n)-\frac{1-l}{2}\hat{\sigma}_{2k,l}(2n)-2n(1-l)\hat{\sigma}_{2k-1,l/2}(2n)\\
&\qquad+2^{2k-1}l(\sigma^*_{2k+1}(n)-n\sigma^*_{2k-1}(n))+(1-l)\Big(\sum_{d|2n}H_{2k}(d)-l\sum_{d|n}H_{2k}(d)\Big).
\end{aligned}$$

This proves (1.3) of Theorem 1.9.

Using (1.2) and (1.3), we get (1.4). Therefore, the proof is completed. ∎

Proof of Theorem 1.12. Following the technique used in [11], we take $f(x) = x^{2k}$ in Proposition 2.3. Then the left hand side of Proposition 2.3 is

$$\begin{aligned}
&\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ 4ax+by=n}} \Big((2a-b)^{2k} - (2a+b)^{2k}\Big)\\
&\quad= \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ 4ax+by=n}} \sum_{s=0}^{2k} \binom{2k}{s}\Big((-1)^s(2a)^{2k-s}b^s - (2a)^{2k-s}b^s\Big)\\
&\quad= -2\sum_{s=0}^{k-1}\binom{2k}{2s+1} \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ 4ax+by=n}} \Big((2a)^{2k-2s-1}b^{2s+1}\Big)\\
&\quad= -2\sum_{s=0}^{k-1}\binom{2k}{2s+1} \sum_{m=1}^{n-1}\sum_{4a|m}(2a)^{2k-2s-1}\sum_{b|n-m} b^{2s+1}\\
&\quad= -2\sum_{s=0}^{k-1}\binom{2k}{2s+1} \sum_{m=1}^{n-1} 2^{2k-2s-1}\sigma_{2k-2s-1}(m/4)\sigma_{2s+1}(n-m)
\end{aligned}$$

The right hand side of Proposition 2.3 is

$$\begin{aligned}
&\frac{1}{2}\sum_{d|n}(1+n/d)d^{2k} - \frac{1}{2}\sum_{d|n/2} d^{2k+1} + \frac{1}{2}\sum_{d|n/4}(1+n/d)(2d)^{2k}\\
&\qquad - \sum_{d|n}\sum_{\substack{x=1\\ x\equiv d(2)}}^{d} x^{2k} - \sum_{d|n/4}\sum_{x=1}^{2d} x^{2k}.
\end{aligned}$$

Set

$$\begin{aligned}
S_1 &= \frac{1}{2}\sum_{d|n}(1+n/d)d^{2k} - \frac{1}{2}\sum_{d|n/2} d^{2k+1} + \frac{1}{2}\sum_{d|n/4}(1+n/d)(2d)^{2k}\\
S_2 &= \sum_{d|n}\sum_{\substack{x=1\\ x\equiv d(2)}}^{d} x^{2k}\\
S_3 &= \sum_{d|n/4}\sum_{x=1}^{2d} x^{2k}.
\end{aligned}$$

By the definition of divisor functions, we see that

$$\begin{aligned}
S_1 = &-\frac{1}{2}\sigma_{2k+1}(n/2) + \frac{1}{2}(\sigma_{2k}(n) + 2^{2k}\sigma_{2k}(n/4))\\
&+ \frac{n}{2}(\sigma_{2k-1}(n) + 2^{2k}\sigma_{2k-1}(n/4)).
\end{aligned} \tag{2.8}$$

Next, we consider

$$\begin{aligned}
S_2 &= \sum_{d|n} \sum_{\substack{x=1 \\ x\equiv d(2)}}^{d} x^{2k} = \sum_{\substack{d|n \\ 2|d}} \sum_{x=1}^{d/2} (2x)^{2k} + \sum_{\substack{d|n \\ 2\nmid d}} \sum_{x=1}^{\frac{d+1}{2}} (2x-1)^{2k} \\
&= \sum_{d|n/2} \sum_{x=1}^{d} 2^{2k} x^{2k} + \sum_{\substack{d|n \\ 2\nmid d}} \sum_{x=1}^{d} x^{2k} - 2^{2k} \sum_{\substack{d|n \\ 2\nmid d}} \sum_{x=1}^{\frac{d+1}{2}-1} x^{2k} \\
&= \frac{2^{2k}}{2k+1} \sum_{j=0}^{2k} \binom{2k+1}{j} B_j \sigma_{2k+1-j}(n/2) + 2^{2k}\sigma_{2k}(n/2) \\
&\quad + \frac{1}{2k+1} \sum_{j=0}^{2k} \binom{2k+1}{j} B_j \sigma^{\#}_{2k+1-j}(n) + \sigma^{\#}_{2k}(n) \\
&\quad - \frac{2^{2k}}{2k+1} \sum_{\substack{d|n \\ 2\nmid d}} \sum_{j=0}^{2k} \binom{2k+1}{j} B_j \left(\frac{d+1}{2}\right)^{2k+1-j}
\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
&\frac{2^{2k}}{2k+1} \sum_{\substack{d|n \\ 2\nmid d}} \sum_{j=0}^{2k} \binom{2k+1}{j} B_j \left(\frac{d+1}{2}\right)^{2k+1-j} \\
&= \sum_{\substack{d|n \\ 2\nmid d}} \sum_{j=0}^{2k} \frac{2^{j-1}}{2k+1} \binom{2k+1}{j} B_j (d+1)^{2k+1-j} \\
&= \sum_{\substack{d|n \\ 2\nmid d}} \sum_{j=0}^{2k} \sum_{i=0}^{2k+1-j} \frac{2^{j-1}}{2k+1} \binom{2k+1}{j} \binom{2k+1-j}{i} B_j d^{2k+1-i-j} \\
&= \frac{1}{2k+1} \sum_{u+v+w=2k+1} 2^{v-1} \binom{2k+1}{u,v,w} B_v \sigma^{\#}_w(n)
\end{aligned} \tag{2.10}$$

Similarly, from (2.9) and (2.10), we obtain

$$\begin{aligned}
S_2 &= \frac{2^{2k}}{2k+1} \sum_{j=0}^{2k} \binom{2k+1}{j} B_j \sigma_{2k+1-j}(n/2) + 2^{2k}\sigma_{2k}(n/2) \\
&\quad + \frac{1}{2k+1} \sum_{j=0}^{2k} \binom{2k+1}{j} B_j \sigma^{\#}_{2k+1-j}(n) + \sigma^{\#}_{2k}(n) \\
&\quad - \frac{1}{2k+1} \sum_{u+v+w=2k+1} 2^{v-1} \binom{2k+1}{u,v,w} B_v \sigma^{\#}_w(n).
\end{aligned} \tag{2.11}$$

Finally, the equation S_3 is equivalent to

$$\begin{aligned}S_3 &= \sum_{d|n/4}\sum_{x=1}^{2d-1} x^{2k} + \sum_{d|n/4}(2d)^{2k}\\ &=\frac{1}{2k+1}\sum_{j=0}^{2k}\binom{2k+1}{j}B_j 2^{2k+1-j}\sigma_{2k+1-j}(n/4) + 2^{2k}\sigma_{2k}(n/4).\end{aligned} \tag{2.12}$$

Hence, by (2.8), (2.11) and (2.12), we obtain

$$\begin{aligned}
&-\frac{1}{2}\sigma_{2k+1}(n/2) + \frac{1}{2}(\sigma_{2k}(n) + 2^{2k}\sigma_{2k}(n/4))\\
&\qquad + \frac{n}{2}(\sigma_{2k-1}(n) + 2^{2k}\sigma_{2k-1}(n/4))\\
&\qquad - \frac{2^{2k}}{2k+1}\sum_{j=0}^{2k}\binom{2k+1}{j}B_j\sigma_{2k+1-j}(n/2) - 2^{2k}\sigma_{2k}(n/2)\\
&\qquad - \frac{1}{2k+1}\sum_{j=0}^{2k}\binom{2k+1}{j}B_j\sigma^{\#}_{2k+1-j}(n) - \sigma^{\#}_{2k}(n)\\
&\qquad + \frac{1}{2k+1}\sum_{u+v+w=2k+1} 2^{v-1}\binom{2k+1}{u,v,w}B_v\sigma^{\#}_w(n)\\
&\qquad - \frac{1}{2k+1}\sum_{j=0}^{2k}\binom{2k+1}{j}B_j 2^{2k+1-j}\sigma_{2k+1-j}(n/4) - 2^{2k}\sigma_{2k}(n/4)\\
&\quad = -\frac{1}{2}\sigma_{2k+1}(n/2) + \frac{1}{2}(\sigma_{2k}(n) - 2^{2k+1}\sigma_{2k}(n/2) - 2^{2k}\sigma_{2k}(n/4))\\
&\qquad + \frac{n}{2}(\sigma_{2k-1}(n) + 2^{2k}\sigma_{2k-1}(n/4)) - \sigma^{\#}_{2k}(n)\\
&\qquad - \frac{2^{2k}}{2k+1}\sum_{j=0}^{2k}\binom{2k+1}{j}B_j\sigma_{2k+1-j}(n/2)\\
&\qquad - \frac{1}{2k+1}\sum_{j=0}^{2k}\binom{2k+1}{j}B_j 2^{2k+1-j}\sigma_{2k+1-j}(n/4)\\
&\qquad - \frac{1}{2k+1}\sum_{j=0}^{2k}\binom{2k+1}{j}B_j\sigma^{\#}_{2k+1-j}(n)\\
&\qquad + \frac{1}{2k+1}\sum_{u+v+w=2k+1} 2^{v-1}\binom{2k+1}{u,v,w}B_v\sigma^{\#}_w(n).
\end{aligned}$$

$\square$

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